**International Journal of Engineering, Science and Mathematics**
Vol. 6 Issue 2, June 2017,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

# CONNECTIVITY IN A FUZZY GRAPH

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**ABSTRACT**

The concept of connectivity and cycle connectivity play an important role in fuzzy graph theory. In this paper cyclic cut vertices, cyclic bridges and cyclically balanced fuzzy graphs are discussed. A cyclic vertex connectivity and cyclic edge connectivity of fuzzy graphs are also discussed. Connectivity of a complement fuzzy graph is analyzed.

**Key Words - Fuzzy Relations, Cyclic Cut Vertices, Cyclic Bridges, Cyclic Vertex Connectivity, Cyclic Edge Connectivity.**

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## 1 INTRODUCTION TO FUZZY GRAPH

Graph theory is proved to be tremendously useful in modeling the essential features of systems with finite components. Graphical models are used to represent telephone network, railway network, communication problems, traffic network etc. Graph theoretic models can sometimes provide a useful structure upon which analytic techniques can be used. A graph is also used to model a relationship between a given set of objects. Each object is represented by a vertex and the relationship between them is represented by an edge if the relationship is unordered and by means of a directed edge if the objects have an ordered relation between them. Relationship among the objects need not always be precisely defined criteria; when we think of an imprecise concept, the fuzziness arises.

In 1965, L.A. Zadeh introduced a mathematical frame work to explain the concept of uncertainty in real life through the publication of a seminar paper. A fuzzy set is defined mathematically by assigning to each possible individual in the universe of discourse a value, representing its grade of membership, which corresponds to the degree, to which that individual is similar or compatible with the concept represented by the fuzzy set. The fuzzy graph introduced by A. Rosenfeld using fuzzy relation, represents the relationship between the objects by precisely indicating the level of the relationship between the objects of the given set. Also he coined many fuzzy analogous graph theoretic concepts like bridge, cut vertex and tree. Fuzzy graphs have many more applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision.

## 2 BASIC DEFINITIONS OF FUZZY GRAPH

**Definition 2.1**

A fuzzy graph G is defined as an ordered pair G=(V,E) where V is set of vertices. A vertex is also called a node or element and E is a set of edges. An edge is an element of the fuzzy set $E : X \times Y \to [0,1]$.

**Definition 2.2**

A fuzzy subset of a non-empty set S is a mapping $\sigma$ : S → [0,1] which assigns to each element 'x' in S a degree of membership, $0 \le \sigma(x) \le 1$.

**Definition 2.3**

A fuzzy relation on S is a fuzzy subset of S × S. A fuzzy relation μ on S is a fuzzy relation on the fuzzy subset $\sigma$ if μ(x, y) $\leq \sigma$ (x) $\wedge$ $\sigma$ (y) for all x, y in S where $\wedge$ stands for minimum. A fuzzy relation on the fuzzy subset $\sigma$ is reflexive if μ(x, x) = $\sigma$ (x) for all x $\in$ S. A fuzzy relation μ on S is said to be symmetric if μ (x, y) = μ (y, x) for all x, y $\in$ S.

$\sigma$ * = supp ( $\sigma$ ) = {u $\in$ S / $\sigma$ (u) > 0}. μ *= supp (μ) = {(u, v) $\in$ S × S / μ (u, v) > 0}.

**Definition 2.4**

A fuzzy graph is a pair G= ( $\sigma$ ,μ) where $\sigma$ is a fuzzy subset of S, μ is a symmetric fuzzy relation on $\sigma$ . The elements of S are called the nodes or vertices of G and the pair of vertices as edges in G. The underlying crisp graph of the fuzzy graph G= ( $\sigma$ ,μ) is denoted as G* : (S, E) where E $\subseteq$ S × S. The crisp graph (S, E) is a special case of the fuzzy graph G with each vertex and edge of (S, E) having degree of membership 1.

**Definition 2.5**

$(\sigma', \mu')$ is a fuzzy sub graph or a partial fuzzy sub graph of ( $\sigma$ ,μ) if $\sigma' \subseteq \sigma$ and $\mu' \subseteq \mu$ ; that is if $\sigma'(u) \leq \sigma(u)$ for every u $\in$ S and $\mu'(e) \leq \mu(e)$ for every e $\in$ E.

**Definition 2.6**

$(\sigma', \mu')$ is a fuzzy spanning sub graph of ( $\sigma$ ,μ) if $\sigma$ ' = $\sigma$ and $\mu' \subseteq \mu$ ; that is if $\sigma$ '(u) = $\sigma$ (u) for every for every u $\in$ S and $\mu'(e) \leq \mu(e)$ for every e $\in$ E.

**Definition 2.7**

For any fuzzy subset $\nu$ of S such that $\nu \subseteq \sigma$ , the fuzzy sub graph of ( $\sigma$ , μ) induced by n is the maximal fuzzy sub graph of ( $\sigma$ ,μ), that has fuzzy vertex set $\nu$ and it is the fuzzy sub graph $(\nu, \tau)$ where $\tau$ (u, v) = $\nu$ (u) $\wedge \nu$ (v) $\wedge \mu$ (u, v) for all u, v in S.

**Definition 2.8**

Given a fuzzy graph G= ( $\sigma$ ,μ), with the underlying set S, the order of G is defined and denoted as $p = \sum_{x \in s} \sigma(x)$ and size of G is defined and denoted as $q = \sum_{x,y \in s} \mu(x, y)$ .

**Definition 2.9**

Let G = ($\sigma$,μ) be a fuzzy graph. The degree of a vertex 'u' is defined as $d(u) = \sum_{\substack{v \neq u \\ v \in s}} \mu(u, v)$ . It is also denoted as dG(u). A fuzzy graph is said to be regular if every vertex is of same degree.

**Definition 2.10**

An edge (x, y) in $\mu$* is an effective edge if $\mu$ (x, y) = $\sigma$ (x) $\wedge$ $\sigma$ (y). A fuzzy graph G is said to be a strong fuzzy graph if $\mu$ (x, y) = $\sigma$ (x) $\wedge$ $\sigma$ (y) for all (x, y) in $\mu$*.

**Definition 2.11**

A fuzzy graph G is said to be a complete fuzzy graph if $\mu$ (x, y) is $\sigma$ (x) $\wedge$ $\sigma$ (y) for all x, y in $\mu$*.

**Definition 2.12**

If $\mu$ (x, y) > 0 then x and y are called neighbours, x and y are said to lie on the edge e = (x, y). A path ρ in a fuzzy graph G = ($\sigma$,μ) is a sequence of distinct nodes $v_0$, $v_1$, $v_2$, …, $v_n$ such that $\mu$ ($v_{i-1}$, $v_i$) > 0, 1≤i≤n. Here 'n' is called the length of the path. The consecutive pairs ($v_{i-1}$, $v_i$) are called arcs of the path.

**Definition 2.13**

If u, v are nodes in G and if they are connected by means of a path then the strength of that path is defined as $\overset{n}{\underset{i=1}{\wedge}} \mu(v_{i-1}, v_i)$ i.e., it is the strength of the weakest arc. If u, v are connected by means of paths of length 'k' then $\mu^k$(u, v) is defined as $\mu^k$(u, v) = sup{ $\mu$ (u, $v_1$) $\wedge$ $\mu$ ($v_1$, $v_2$) $\wedge$ $\mu$ ($v_2$, $v_3$) … $\wedge$ $\mu$ ($v_{k-1}$, v) / u, $v_1$, $v_2$, …, $v_{k-1}$, v $\in$ S}. If u, v $\in$ S the strength of connectedness between u and v is, $\mu^\infty$(u, v) = sup{ $\mu^k$ (u, v) / k = 1, 2, 3, …}.

**Definition 2.14**

A fuzzy graph G is connected if $\mu^{\infty}(u, v) > 0$ for all u, $v \in \sigma^*$. An arc (x, y) is said to be a strong arc if $\mu(x, y) \geq \mu^{\infty}(x, y)$. A node x, is said to be an isolated node if $\mu(x, y) = 0 \ \forall\, y \neq x$.

**Definition 2.15**

G = $(\sigma, \mu)$ is a fuzzy cycle iff $(\sigma^*, \mu^*)$ is a cycle and there does not exist a unique $(x, y) \in \mu^*$ such that $\mu(x, y) = \wedge \{ \mu(u, v) / (u, v) \in \mu^* \}$.

**Definition 2.16**

Let G = $(\sigma, \mu)$ be a fuzzy graph. The complement of G is defined as $G^c = (\sigma^c, \mu^c)$ where $\mu^c(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \ \forall\, x, y \in S$.

**Definition 2.17**

The $\mu$ -complement of G is denoted as $G^{\mu} = (\sigma^{\mu}, \mu^{\mu})$ where $\sigma^{\mu} = \sigma$ and $\mu^{\mu}(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ if $\mu(u, v) > 0$.

**Definition 2.18**

The busy value of a node 'v' in G is D(v) = $\sum_i \sigma(v) \wedge \sigma(v_i)$ where $v_i$ are the neighbours of v and the busy value of G is D(G) = $\sum_i D(v_i)$ where $v_i$ are the nodes of G.

**Definition 2.19**

A node in G is a busy node if $\sigma(v) \leq d(v)$, otherwise it is called a free node.

**Definition 2.20**

A node v of a fuzzy graph G is said to be

i. A partial free node if it is a free node in both G and $G^{\mu}$.

ii. Fully free node if it is free in G but busy in $G^{\mu}$.

iii. Partial busy node if it is a busy node in both G and $G^{\mu}$.

iv. Fully busy node if it is busy in G but free in $G^{\mu}$

**Definition 2.21**

Two nodes of a fuzzy graph are said to be fuzzy independent if there is no strong arc between them.

**Definition 2.22**

A subset S' of S is said to be fuzzy independent if any two nodes of S' are fuzzy independent.

**Definition 2.23**

A fuzzy graph G is said to be fuzzy bipartite if the node set S can be partitioned into two subsets $S_1$, $S_2$ such that $S_1$ and $S_2$ are fuzzy independent sets. These sets are called fuzzy bipartition of S.

**Definition 2.24**

A fuzzy matrix is the matrix whose elements are taking their values from [0, 1].

**Definition 2.25**

A fuzzy graph that has no cycles is called a cyclic or a forest. A connected forest is called a fuzzy tree. It is also denoted as f-tree.

**Definition 2.26**

A cyclic vertex cut of a fuzzy graph G=($\sigma$, μ) is a set of vertices $X \subseteq \sigma^*$ such that CC(G-X)<CC(G) provided CC(G)>0 where CC(G) is the cycle connectivity of G.

**Definition 2.27**

Let X be a cyclic vertex cut of G. The strong weight of X is defined as $S_c(X) = \sum_{x \in X} \mu(x, y)$ where $\mu(x, y)$ is the minimum of weights of strong edges incident on X.

**Definition 2.28**

Cyclic vertex connectivity of a fuzzy graph G, denoted by $k_c(G)$, is the minimum if the cyclic strong weights of cyclic vertex cuts in G.

**Definition 2.29**

A cyclic edge cut of a fuzzy graph G=( $\sigma$ , μ) is a set of edges $Y \subseteq \mu^*$ such that CC(G-Y)<CC(G), provided CC(G)>0, where CC(G) is the cyclic connectivity in G.

**Definition 2.30**

Let G=( $\sigma$ , μ) be a fuzzy graph. The strong weight of a cyclic edge cut Y of G is defined as $S_c^{'}(Y) = \sum_{e_i \in \mu^*} \mu(e_i)$, where $e_i$ is a strong edge of Y.

**Definition 2.31**

Cyclic edge connectivity of a fuzzy graph G denoted by $k_c^{'}(G)$ is the minimum of the strong weights of cyclic edge cuts in G.

**Definition 2.32**

A fuzzy graph is cyclically balanced if it has no cyclic fuzzy cutvertices and cyclic fuzzy cutbridges.

**3.CONNECTIVITY IN A FUZZY GRAPH**

**3.1 CYCLE CONNECTIVITY IN FUZZY GRAPHS**

**Theorem 3.1.1**

A fuzzy graph G is a fuzzy tree if and only if CC(G) = 0.

**Proof**

If G is an f-tree, then CGu,v = 0 for every pair of nodes u and v in G. Hence it follows that CC(G) = 0.

Conversely suppose that CC(G) = 0. Hence CGu,v = 0 for every pair of nodes in G. That is G has no strong cycles. Hence G has no fuzzy cycles and hence it follows that G is an f-tree.

**Proposition 3.1.2**

The cycle connectivity of a fuzzy cycle G is the strength of G.

**Proof**

Proof follows from the fact that any fuzzy cycle is a strong cycle.

**Theorem 3.1.3**

Let G be a complete fuzzy graph with nodes $v_1, v_2, ...., v_n$ such that $\sigma(v_i) = t_i$ and $t_1 \leq t_2 \leq ......\leq t_{n-2} \leq t_{n-1} \leq t_n$. Then $CC(G) = t_{n-2}$.

**Proof**

Assume the conditions of the Theorem. Since any three nodes of G are adjacent, any three nodes are in a 3-cycle. Also all arcs in a complete fuzzy graph are strong . Thus to find the maximum strength of cycles in G, it is sufficient to find the maximum strength of all 3-cycles in G. For consider a 4-cycle C = abcda in G (case of n-cycle is similar). Since G is complete, there exists parts of two 3-cycles in C, namely $C_1$ = abca and $C_2$ = acda. Let the strength s(C) = t. For all edges (x, y) in C, $\mu(x, y) \geq t$. In particular $\mu(a, b) \geq t$ and $\mu(b, c) \geq t$. Since G is a complete fuzzy graph, G has no $\delta$ arcs. Thus $\mu(a, c) \geq Min\{\mu(a, b),\mu(b, c)\}\geq t$. That is $\mu(a, c) \geq t$. Suppose $\mu(a, c) = t$, then $s(C_1) = s(C_2) = s(C) = t$.

Suppose $\mu(a, c) > t$, then since s(C) = t, atleast one of $C_1$ or $C_2$ will have strength equal to t.
In either case, $s(C) = Min\{s(C_1), s(C_2)\}$ Thus the strength of a 4-cycle is nothing but the strength of a 3-cycle in G. Among all 3-cycles, the 3-cycle formed by three nodes with maximum node strength will have the maximum strength. Thus the cycle $C = v_{n-2}v_{n-1}v_nv_{n-2}$ is a cycle with maximum strength in G. Also Strength of $C = t_{n-2}\wedge t_{n-1}\wedge t_n = t_{n-2}$ where ^ stands for the minimum. Thus $CC(G) = t_{n-2}$.

**Proposition 3.1.4**

In a fuzzy graph, if arc (u, v) is a cyclic bridge, then both u and v are cyclic cutnodes.

**Proof**

Let $G(\sigma, \mu)$ be a fuzzy graph and (u, v) be a cyclic bridge in G. Then $CC(G - (u, v)) < CC(G)$. Hence $CC(G - u) \leq CC(G - (u, v) < CC(G)$ and $CC(G - v) \leq CC(G - (u, v) < CC(G)$. Thus u and v are cyclic cutnodes.

**Proposition 3.1.5**

Let G be a fuzzy graph such that G* is a cycle. Then,

(a) G has no cyclic cutnodes or cyclic bridges if G is a fuzzy tree.

(b) All arcs in G are cyclic bridges and all nodes in G are cyclic cutnodes if G is a strong cycle.

**Proof**

(a) Follows from the fact that a fuzzy tree has no strong cycles.

(b) If G is a strong cycle, then CC(G) = strength of G. The removal of any arc or node will reduce its cycle connectivity 0.

**Theorem 3.1.6**

Let G = ($\sigma$, μ) be a complete fuzzy graph with $|\sigma^*| \geq 4$. Let $v_1, v_2, ..., v_n \in \sigma^*$ and $\sigma(v_i) = c_i$ for i = 1, 2, ..., n and $c_1 \leq c_2 \leq ... \leq c_n$. Then G has a cyclic cutnode (or cyclic bridge) if and only if $c_{n-3} < c_{n-2}$. Further there exists three cyclic cutnodes ( or cyclic bridges) in a complete fuzzy graph (if exists).

**Proof**

Let $v_1, v_2, ..., v_n \in \sigma^*$ and $\sigma(v_i) = c_i$ for i = 1, 2, ..., n and $c_1 \leq c_2 \leq ... \leq c_n$. Suppose that G has a cyclic cutnode u(say). Then CC(G − u) < CC(G). That is u belongs to a unique cycle C with $\alpha$ = strength of C > strength of C′ for any other cycle C′ in G. Since $c_1 \leq c_2 \leq ... \leq c_n$, it follows that the strength of the cycle $v_{n-2}v_{n-1}v_n$ is $\alpha$. Hence $u \in \{v_{n-2}, v_{n-1}, v_n\}$...(1).

To prove $c_{n-3} < c_{n-2}$. Suppose not. That is $c_{n-3} = c_{n-2}$.

Then $C_1 = v_n v_{n-1} v_{n-2}$ and $C_2 = v_n v_{n-1} v_{n-3}$ have the same strength and hence the removal of $v_{n-2}$, $v_{n-1}$ or $v_n$ will not reduce CC(G) which is a contradiction to (1). Hence $c_{n-3} < c_{n-2}$.

Conversely suppose that $c_{n-3} < c_{n-2}$. To prove G has a cyclic cutnode. Since $c_n \geq c_{n-1} \geq c_{n-2}$ and $c_{n-2} > c_{n-3}$, all cycles of G have strength less than that of strength of $v_n v_{n-1} v_{n-2}$. Hence the deletion of $v_n$, $v_{n-1}$ or $v_{n-2}$ will reduce the cycle connectivity of G. Hence $v_n$, $v_{n-1}$ and $v_{n-2}$ are cyclic cutnodes of G.

**Theorem 3.1.7**

For a complete fuzzy graph G, $k_c(G) \leq k(G)$.

**Proof**

Given a complete fuzzy graph G with vertices $v_1, v_2, ..., v_n$ such that $d_s(v_1) \leq d_s(v_2) \leq \ldots \leq d_s(v_n)$. Let $v_1$ be a vertex such that $d_s(v_1) = \delta_s(G)$.

**Case I** If $v_1$ is a cyclic cutvertex.

Here, V={$v_1$} is a cyclic cut set of G. Therefore,

$$S_c(V) = \min\{\mu(v_1, v_i)\} \text{ for i=\{2,....n\}}$$

$$\leq \sum_i \mu(v_1, v_i)$$

$$= \delta_s(G)$$

Now, since $k_c(G)=\min\{S_c(V)\}$, where V is a cyclic cutest of G. we have

$$k_c(G) \leq S_c(V) \leq \delta_s(G) = k(G)$$

**Case II** If $v_1$ is not a cyclic cutvertex.

Let F={ $u_1$, $u_2$,....$u_t$} be a cyclic cut set such that $S_c(F)= k_c(G)$. Now

$k_c(G)= S_c(F)$

$$= \sum_{i=1}^{t} \min\{\mu(u_i, u_j)\}, \forall u_i, u_j \in \sigma^* \; for \; j \neq i, \; j = 1,2,...n$$

$$= \sum_{i=1}^{t} \min\{\mu(u_i, u_j)\}$$

$$\leq d_s(v_1)$$

$$= \delta_s(G)$$

$$= k(G)$$

**Corollary 3.1.8**

A vertex in a fuzzy graph is cyclic cutvertex if and only if it is a common vertex of all strong cycles with maximum strength.

**Proof**

Let G be a fuzzy graph. Let w be a cyclic cutvertex of G. Then CC(G-w)<CC(G), i.e., Max{S(C), where C is a strong cycle in G-w}<Max {s(C′), where C′ is a strong cycle in G}. Therefore all strong cycles in G with maximum strength is removed by the deletion of w. Hence w is common vertex of all strong cycles with maximum strength. Hence it will results in the reduction of cycle connectivity of G. Thus w is a cyclic cutvertex of G.

**Theorem 3.1.9**

Let G = ($\sigma$ , μ) be a fuzzy graph. Then no cyclic cutvertex is a fuzzy endvertex of G.

**Proof**

Let G = ($\sigma$ , μ) be a fuzzy graph. Let w be a cyclic cutvertex of G. Then w lies on a strong cycle with maximum strength in G. Clearly w has atleast two strong neighbors in G. Hence w cannot be a fuzzy endvertex of G.

Conversely, if w is a fuzzy endvertex of G with $|N_s(w)| =1$, where $N_s(w)$ be the neighboring set of w, then w cannot lie on a strong cycle in G. Implies w is not a cyclic cutvertex of fuzzy graph G.

**Theorem 3.1.10**

Let G = ($\sigma$ , μ) be a complete fuzzy graph with $|\sigma^*| \geq 4$. Suppose $v_1, v_2, ..., v_n \in \sigma^*$ and $\sigma(v_i) = c_i$ for i = 1, 2, ..., n and $c_1 \leq c_2 \leq ... \leq c_n$. Then G is cyclically balanced if and only if $c_{n-3}= c_{n-2}$.

**Proof**

Let $v_1, v_2, ..., v_n \in \sigma^*$ and $\sigma(v_i) = c_i$ for i = 1, 2, ..., n and $c_1 \leq c_2 \leq ... \leq c_n$.If possible, suppose that G is cyclically balanced. To prove that $c_{n-3}= c_{n-2}$ . Suppose not, that is $c_{n-3}< c_{n-2}$. Since $c_{n-2} \leq c_{n-1} \leq c_n$ and $c_{n-3}< c_{n-2}$, all cycles of G have strength less than that of strength $v_n v_{n-1} v_{n-2} v_n$. Hence the deletion of any of the three vertices $v_n$ ,$v_{n-1}$ or $v_{n-2}$ reduces the cycle connectivity of G. Hence $v_n$ ,$v_{n-1}$ and $v_{n-2}$ are cyclic cutvertices of G, which is a contradiction to the fact that G is cyclically balanced.

Conversely, suppose that $c_{n-3}= c_{n-2}$ .Then $C_1= v_nv_{n-1}v_{n-2}v_n$ and $C_2= v_nv_{n-1}v_{n-3}v_n$ have the same strength and hence the removal of $v_n$ ,$v_{n-1}$ or $v_{n-2}$ will not reduce the cyclic connectivity of G. That is , there does not exist any cyclic fuzzy cutvertex in G.

Hence the fuzzy graph G = ($\sigma$ , μ) is cyclically balanced.

**Theorem 3.1.11**

Let G be a complete fuzzy graph. G is cyclically balanced if there exist a $K_4$ as a sub graph of G, in which every cycle is of equal maximum strength.

**Proof**

Let G be a complete fuzzy graph with $|\sigma^*| \geq 4$. Let $v_1, v_2, ..., v_n \in \sigma^*$ and $\sigma(v_i) = c_i$ for i = 1, 2, ..., n and $c_1 \leq c_2 \leq ... \leq c_n$. Let $K_4$ be a fuzzy sub graph of G with vertex set $\{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ such that $c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n$. Suppose all the strong cycles in $K_4$ are of equal maximum strength. This happens only when $c_{n-3} = c_{n-2}$. Then by theorem 3.1.10 is cyclically balanced.

**Theorem 3.1.12**

Let G = ($\sigma$, μ) be a complete fuzzy graph and $v \in \sigma^*$ such that $d_s(v) = \Delta_s(G)$. If v lies on a strong cycle C, then S(C)=CC(G).

**Proof**

Let G = ($\sigma$, μ) be a complete fuzzy graph and $v \in \sigma^*$ such that $d_s(v) = \Delta_s(G)$.

Let $v_1, v_2, ..., v_n \in \sigma^*$ and $\sigma(v_i) = c_i$ for i = 1, 2, ..., n and $c_1 \leq c_2 \leq ... \leq c_n$. Since $c_{n-2} \leq c_{n-1} \leq c_n$ and $c_{n-3} < c_{n-2}$ , all cycles of G have strength less than that of the strength of $v_n v_{n-1} v_{n-2} v_n$. First to prove that for $v_i$, i=1,2,….,n-2. $d_s(v_i) < d_s(v_n)$.

$$\begin{aligned} d_s(v_i) &= \sum_{j=1, j\neq i}^{n} \min(c_i, c_j) \\ &= c_1 + c_2 + ... + (n-i)c_i,\ i = 1,2,...n-2 \\ &= \sum c_i,\ i = 1,2,...n-1 \\ &= d_s(v_n). \end{aligned}$$

Also

$$\begin{aligned} d_s(v_n) &= (c_{n-1} \wedge c_n) + \sum_{i=1}^{n-2} (c_n \wedge c_i) \\ &= (c_{n-1} \wedge c_n) + \sum_{i=1}^{n-2} (c_{n-1} \wedge c_i) \\ &= d_s(v_{n-1}) \\ &= \sum_{i=1}^{n-1} c_i \\ &= \Delta_s(G) \end{aligned}$$

Therefore, $v_n$ belongs to the strong cycle $c_n c_{n-1} c_{n-2} c_n$, where strength of this cycle is equal to the cycle connectivity of graph G.

**Theorem 3.1.13**

For a complete fuzzy graph, $k_c(G) \leq k_c^{'}(G) \leq \delta_s(G)$.

**Proof**

Consider all cycles, with strength equal to cycle connectivity of $G = (\sigma, \mu)$.

Let X={$e_1$,$e_2$,…$e_n$}, where $e_i$=($u_i$,$v_i$) be one of the edges in each such cycles. Then X from a cyclic fuzzy edge cut of G. Let $S_c$(X) be the cyclic strong weight of X. Then, according to the definition of cyclic fuzzy edge connectivity,

$k_c^{'}(G) \leq S_c(X)$.

Let Y={$v_1$,$v_2$,….,$v_n$} where $v_i$ is of one of the end vertices of $e_i$. Then Y forms a cyclic fuzzy vertex cut of G. Let $S_c$(Y) is the cyclic strong weight of Y. Then

$S_c(Y). \leq k_c^{'}(G)$

Hence

$k_c(G) \leq S_c(Y)$.

From the above equation

$$\begin{aligned} k_c(G) &\leq S_c(Y) \\ &\leq k_c^{'}(G) \\ &\leq \delta_s(G) \end{aligned}$$

By theorem 3.1.7

Hence, $k_c(G) \leq k_c^{'}(G) \leq \delta_s(G)$.

**Theorem 3.1.14**

A fuzzy graph G = ($\sigma$ , μ) with $|\sigma^*| \geq 6$ is cyclically balanced if there exist two disjoint cycles $C_1$ and $C_2$ such that S($C_1$)=S($C_2$)=CC(G).

**Proof**

Let G = ($\sigma$ , μ) be a fuzzy graph with $|\sigma^*| \geq 6$ and cycle connectivity of G is equal to CC(G). Let $C_1$ and $C_2$ two disjoint cycles in G such that S($C_1$)=S($C_2$)=CC(G).

Suppose u is a vertex not in $V(C_1 \cup C_2)$. Then the deletion of u will not effect the cycle connectivity of G-{u}. If the vertex $u \in V(C_1)$ and if u is deleted, then the cycle connectivity remains the same, since there exist another cycle $C_2$ with strength of $C_2$ equal to the cycle connectivity of the fuzzy graph. Similarly if $u \in V(C_2)$, u is not a cyclic cutvertex.

Suppose $(u,v) \in E$ but $(u,v) \notin E(C_1 \cup C_2)$, clearly (u,v) will not reduce the cycle connectivity of the fuzzy graph. If (u,v) is an edge either on $C_1$ or on $C_2$, then the removal of (u,v) from any one of these cycles will not effect the cycle connectivity of G. Hence (u,v) is not a cyclic fuzzy bridge.

**Theorem 3.1.15**

For $|\sigma^*| \geq 4$, There is a connected cyclically balanced fuzzy graph.

**Proof**

For $|\sigma^*| = 4$ and 5, have cyclically balanced fuzzy graphs from the definition 2.2.32. For $|\sigma^*| \geq 6$ prove the result by induction. For $|\sigma^*| = 6$, let $v_1, v_2, \ldots, v_6$ be the 6 vertices. Construct two edge disjoint cycles $C_1 = v_1v_2v_3v_1$ and $C_2 = v_4v_5v_6v_4$ with maximum strength.

Join each pair of vertices from the two cycles and make the graph complete. Then the removal of an edge or a vertex will not reduce the cycle connectivity of G. So the newly obtained fuzzy graph is cyclically balanced.

Assume that the result is true for $|\sigma^*| = k$. let $G_k$ be a cyclically balanced fuzzy graph with k vertices. Then there exist two disjoint cycles of maximum strength in $G_k$.

Let $G_{k+1}$ be the fuzzy graph obtained from $G_k$ by adding one more vertex u. Make the fuzzy graph complete by connecting all vertices of $G_k$ with u. Also assign a membership value to all newly joined edges, which less than or equal to the cycle connectivity of $G_k$.

In this case, if we remove the vertex u, then the cycle connectivity of $G_k$ remains the same. In a similar way, the removal of any edge incident on k+1$^{th}$ vertex u will not change the cycle connectivity of G. Therefore cycle connectivity of $G_{k+1}$ remains the same. Hence $G_{k+1}$ is cyclically balanced.

**3.2 CONNECTIVITY IN $G^c$**

**Preposition 3.2.1**

Let $G = (\sigma, \mu)$ be connected fuzzy graph with no m-strong arcs then $G^c$ is connected.

**Proof**

The fuzzy graph G is connected and contain no m-strong arcs. Suppose u ,v be two arbitrary nodes of $G^c$. Then they are also nodes of G. Since G is connected there exist a path between u and v in G. Let this path be P. Then $P = (u_0, u_1)(u_1, u_2)...(u_{n-1}, u_n)$ where $\mu(u_{i-1}, u_i) > 0 \; \forall \; i.$

Since G contain no m-strong arcs, $\mu^c(u_{i-1}, u_i) > 0 \; \forall \; i.$

Hence P will be a (u,v) path in $G^c$ also. Therefore $G^c$ is connected.

**Theorem 3.2.2**

Let $G = (\sigma, \mu)$ be a fuzzy graph. G and $G^c$ are connected if and only if G contains at least one connected spanning fuzzy subgraph with no m-strong arcs.

**Proof**

Suppose that G contains a spanning subgraph H that is connected, having no m-strong arcs. Since H contain no m-strong arcs and is connected using proposition-3.2.1, $H^c$ will be a connected spanning fuzzy subgraph of $G^c$ and thus $G^c$ is also connected.

Conversely assume that G and $G^c$ are connected. Have to find a connected spanning subgraph of G that contain no m-strong arcs.

Let H be an arbitrary connected spanning subgraph of G. If H contain no m-strong arcs then H is the required subgraph. Suppose H contain one m-strong arc say (u,v). Then arc (u,v) will not be present in $G^c$. Since $G^c$ is connected there will exist a u-v path in $G^c$. Let this path be $P_1$. Let $P_1 = (u_1,u_2)(u_2,u_3)...(u_{n-1},u_n)$, where $u_1 = u$ and $u_n = v$.

If all the arcs of $P_1$ are present in G then H-(u,v) together with $P_1$ will be the required spanning subgraph. If not, there exist at least one arc say $(u_1, v_1)$ in $P_1$ which is not in G. Since G is connected we can replace $(u_1, v_1)$ by another $u_1$-$v_1$ path in G. Let this path be $P_2$. If $P_2$ contain no m-strong arcs then H-(u,v) - $(u_1, v_1)$ together with $P_1$ and $P_2$ will be the required spanning subgraph. If $P_2$ contain an m-strong arc then this arc will not be present in $G^c$. Then replace this arc by a path connecting the corresponding vertices in $G^c$ and proceed as above and since G contain only finite number of arcs finally we will get a spanning subgraph of that contain no m-strong arcs.

If more than one m-strong arc is present in H, then the above procedure can be repeated for all other m-strong arcs of H to get the required spanning subgraph of G.

**Corollory 3.2.3**

Let $G = (\sigma, \mu)$ be a fuzzy graph. G and $G^c$ are connected if and only if G contains at least one fuzzy spanning tree having no m- strong arcs.

**Proof**

Using theorem 3.2.2 we will get a connected fuzzy spanning subgraph of G which contain no m -strong arcs. The maximum spanning fuzzy tree of this subgraph will be a spanning fuzzy tree of G that contain no m - strong arcs.

## 3.3 COMPLEMENT OF FUZZY CYCLES

**Theorem 3.3.1**

Let $G = (\sigma, \mu)$ be a fuzzy graph such that $G^*$ is a cycle with more than 5 vertices. Then $(G^*)^c$ cannot be a cycle.

**Proof**

Given $G^*$ is a cycle having n nodes where $n \geq 6$. Then $G^*$ will have exactly n arcs. Since all the nodes of G are also present in $G^c$ number of nodes of $G^c$ is n . Let the nodes of G and $G^c$ be $v_1, v_2, ... v_n$.

Then $G^c$ must contain at least the following edges.

$(v_1, v_3),(v_1, v_4) \ldots (v_1, v_n)$ ; $(v_2 ; v_4),(v_2, v_5) \ldots (v_2, v_n)$; $(v_3, v_5)$, $(v_3, v_6) \ldots (v_3, v_n)$

Since $n \geq 6$ the total number of edges in $G^c$ will be greater than n. Thus $G^c$ will not be a cycle.

**Corollory 3.3.2**

Let G be fuzzy cycle with 6 or more nodes. Then $G^c$ will not be fuzzy cycle.

**Proof**

Given G be fuzzy cycle with 6 or more nodes. All the nodes of G is also node of $G^c$.

Then $G^c$ must contain at least the following edges.

$(v_1, v_3),(v_1, v_4) \ldots (v_1, v_n)$ ; $(v_2 ; v_4),(v_2, v_5) \ldots (v_2, v_n)$; $(v_3, v_5)$, $(v_3, v_6) \ldots (v_3, v_n)$

Since $n \geq 6$ the total number of edges in $G^c$ will be greater than n. Thus $G^c$ will not be a cycle.

## 4. CONCLUSION

The fuzzy graph represents the relationship between the objects of the given set. Fuzzy graphs have many more applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. In this paper Criterion for connectivity of a fuzzy graph is analyzed. The concept of connectivity and cycle connectivity play an important role in fuzzy graph theory. Cyclic vertex connectivity and cyclic edge connectivity of fuzzy graphs are also discussed. The complement of a fuzzy cycle is discussed.

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